Reducing decoherence in quantum computer memory with all quantum bits coupling to the same environment

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Abstract

Decoherence in quantum computer memory due to the inevitable coupling to the external environment is examined. We take the assumption that all quantum bits (qubits) interact with the same environment rather than the assumption of separate environments for different qubits. It is found that the qubits are decohered collectively. For some kinds of entangled input states, no decoherence occurs at all in the memory even if the qubits are interacting with the environment. Based on this phenomenon, a scheme is proposed for reducing the collective decoherence. We also discuss possible implications of this decoherence model for quantum measurements.

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Quantum computers have recently raised a lot of interest [1,2]. In quantum computers, the contents of the memory cells are in a superposition of different states and the computer performs deterministic unitary transformations on the quantum states of these memory cells [3]. A two-state memory cell, which may be a spin-$\frac{1}{2}$ electron or a two-level atom, is called a quantum bit, or qubit [4]. It has been argued that quantum computers can solve certain problems much more efficiently than the classical computers [5-8]. An impressive example is, Shor [8] has shown a quantum computer could solve the problem of finding factors of a large number N in a time which is a polynomial function of the length L (number of bits) of the number. However, it is not yet clear whether quantum computers are feasible to build. Decoherence is one of the major obstacles to realizing quantum computation. It has been found that decoherence in quantum computer memory can not be neglected if the qubits interact with the external environment [9,10]. To reduce this decoherence, some strategies, such as the quantum error-correction [11-17] and the purification of noisy entanglement [18,19], have been proposed.

In the previous analysis of decoherence [9,10], an important assumption has been taken. This is that the qubits are assumed to couple independently to separate environments. Independent decoherence is an ideal case. As has been pointed out by Ekert and Lloyd [9], there is another practical circumstance, in which the qubits interact with the same environment. The interaction with the same environment will result in cooperative decoherence for the qubits. Plama et al. [20] provided the first step in studying the cooperative decoherence. They started by considering a system of two qubits and extended the result to include a register of L qubits. In this paper, we propose an alternative simple approach for studying the collective decoherence and correct an error in the calculation of Ref. [20]. We

consider a system of L qubits interacting with the same environment. It is found that the qubits are decohered collectively. This is compared with the independent decoherence of the qubits when they interact with separate environments. Because of the collective decoherence, for some kinds of input states (called the coherence-preserving states), no decoherence occurs at all even if the qubits are interacting with the environment. Based on this phenomenon, a simple scheme is proposed for reducing the collective decoherence. The coherence-preserving states are intimately related to the concept of the point basis introduced in the theory of quantum measurements [21]. We also discuss possible implications of this decoherence model for quantum measurements.

Now we consider the decoherence model—L qubits in the memory jointly coupled to the same environment. The l qubit can be described by the Pauli operator $\overrightarrow{\sigma_l}$. The environment is modelled by a bath of oscillators. We consider the decoherence resulting from the dephasing process. The Hamiltonian describing the phase damping has the form [22]

$$H = \hbar \left\{ \int d\omega \left[\sum_{l=1}^{L} \kappa(\omega) \left(a_\omega + a_\omega^+ \right) \sigma_l^z + \omega a_\omega^+ a_\omega \right] \right\}, \tag{1}$$

where L indicates number of the qubits. We have supposed that the coupling constants $\kappa(\omega)$ to the environment are the same for all the qubits. In the Hamiltonian (1), σ_l^z ($l = 1, 2, \cdots, L$) are conservative operators, so the dynamical equations for the operators can be easily solved. The Heisenberg equation for the bath operator is

$$i \dot{a}_\omega = \omega a_\omega + \sum_{l=1}^{L} \kappa(\omega) \sigma_l^z. \tag{2}$$

It has the solution

$$a_\omega(t) = a_\omega(0) e^{-i\omega t} + \sum_{l=1}^{L} \kappa(\omega) \sigma_l^z \frac{e^{-i\omega t} - 1}{\omega}. \tag{3}$$

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To determine the magnitude of decoherence, we need know the evolution of the reduced density of the qubits. This problem can be solved by using the operator representation of the density operator [23]. Let $\rho_{-1,-1} = \frac{1}{2}\left(I + \sigma^z\right)$, $\rho_{1,1} = \frac{1}{2}\left(I - \sigma^z\right)$, $\rho_{-1,1} = \sigma^+$, $\rho_{1,-1} = \sigma^-$, and

$$\rho_{\{i_l, j_l\}} = \rho_{i_1, j_1} \otimes \rho_{i_2, j_2} \otimes \cdots \otimes \rho_{i_L, j_L}, \tag{4}$$

where I is the unit operator, and possible values for i_l and j_l $(l = 1, 2, \cdots, L)$ are 1 or -1. All the operators defined by Eq. (4) are expressed by Pauli's operators. Obviously, they make a complete set. The initial density operator of the qubits can be expanded into the set of operators $\rho_{\{i_l, j_l\}}$. Suppose the environment is initially in thermal equilibrium. The total density operator $\rho(0)$ is then expressed as

$$\rho(0) = \rho_s(0) \otimes \rho_{env}(0)$$
$$= \sum_{\{i_l, j_l\}} c_{\{i_l, j_l\}} \rho_{\{i_l, j_l\}}(0) \otimes \prod_\omega \int d^2\alpha_\omega \frac{1}{\pi \langle N_\omega \rangle} \exp\left(-\frac{|\alpha_\omega|^2}{\langle N_\omega \rangle}\right) \left(|\alpha_\omega\rangle \langle \alpha_\omega|\right)_0, \tag{5}$$

where the subscripts s and env denote the system (qubits) and the environment, respectively. $\langle N_\omega \rangle$ is the mean photon number of the bath mode ω

$$\langle N_\omega \rangle = \frac{1}{e^{\hbar\omega/k_B T} - 1}. \tag{6}$$

In the Schrödinger picture, the density operator obeys Von Neumann's equation which differs from Heisenberg's equation by an overall sign. Therefore, the density operator (in the Schrödinger picture) may be treated as an ordinary operator (in the Heisenberg picture) evolving backwards in time. We thus have

$$\rho(t) = \sum_{\{i_l, j_l\}} c_{\{i_l, j_l\}} \rho_{\{i_l, j_l\}}(-t) \otimes \prod_\omega \int d^2\alpha_\omega \frac{1}{\pi \langle N_\omega \rangle} \exp\left(-\frac{|\alpha_\omega|^2}{\langle N_\omega \rangle}\right) \left(|\alpha_\omega\rangle \langle \alpha_\omega|\right)_{-t}, \tag{7}$$

where $\rho_{\{i_l,j_l\}}$ and $|\alpha_\omega\rangle\langle\alpha_\omega|$ are treated as ordinary operators. From Eq. (3), we know

$$(|\alpha_\omega\rangle\langle\alpha_\omega|)_{-t} = \left| \alpha_\omega e^{-i\omega t} + \sum_{l=1}^{L} \kappa(\omega)\sigma_l^z \frac{e^{-i\omega t}-1}{\omega} \right\rangle \left\langle \alpha_\omega e^{-i\omega t} + \sum_{l=1}^{L} \kappa(\omega)\sigma_l^z \frac{e^{-i\omega t}-1}{\omega} \right|_0. \tag{8}$$

So the reduced density operator of the qubits at time t can be expressed as

$$\rho_s(t) = Tr_{env}(\rho(t)) = \sum_{\{i_l,j_l\}} c_{\{i_l,j_l\}} \otimes \int \langle\{\alpha_\omega\}| \rho_{\{i_l,j_l\}}(-t) |\{\alpha_\omega\}\rangle \prod_\omega \left[P_\omega(\alpha_\omega,t)d^2\alpha_\omega \right], \tag{9}$$

where $|\{\alpha_\omega\}\rangle = \prod_\omega \otimes |\alpha_\omega\rangle$ indicates the co-eigenstates of all the bath operators $a_\omega(0)$ and

$$P_\omega(\alpha_\omega,t) = \frac{1}{\pi\langle N_\omega\rangle} \exp\left[-\frac{1}{\langle N_\omega\rangle} \left| \alpha_\omega e^{i\omega t} + \sum_{l=1}^{L} \kappa(\omega)\sigma_l^z \frac{e^{i\omega t}-1}{\omega} \right|^2 \right]. \tag{10}$$

Now we need to obtain the diagonal matrix elements of the operator $\rho_{\{i_l,j_l\}}(-t)$ in the bath coherent representation. From the Hamiltonian (1), the Heisenberg equation for the operator $\rho_{\{i_l,j_l\}}$ is

$$i\,\dot{\rho}_{\{i_l,j_l\}} = \left[\sum_{l=1}^{L}(i_l - j_l) \right] \left[\int d\omega\kappa(\omega)(a_\omega + a_\omega^+) \right] \rho_{\{i_l,j_l\}}$$

$$= \left[\sum_{l=1}^{L}(i_l - j_l) \right] \left[\int d\omega\kappa(\omega) \left(\rho_{\{i_l,j_l\}}a_\omega(0)e^{-i\omega t} + a_\omega^+(0)e^{i\omega t}\rho_{\{i_l,j_l\}} \right) \right]$$

$$- \int d\omega\kappa^2(\omega) \left\{ \frac{\cos(\omega t)-1}{\omega} \left[\left(\sum_{l=1}^{L} i_l \right)^2 - \left(\sum_{l=1}^{L} j_l \right)^2 \right] + i\frac{\sin(\omega t)}{\omega} \left[\sum_{l=1}^{L}(i_l - j_l) \right]^2 \right\} \rho_{\{i_l,j_l\}}. \tag{11}$$

In the derivation, the following relations are used:

$$\rho_{\{i_l,j_l\}}\sigma_l^z = -j_l\rho_{\{i_l,j_l\}}, \tag{12}$$

$$\sigma_l^z\rho_{\{i_l,j_l\}} = -i_l\rho_{\{i_l,j_l\}}. \tag{13}$$

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The solution of Eq. (11) is

$$\rho_{\{i_l,j_l\}}(t) = \exp\left\{i \int d\omega \kappa^2(\omega) \left\{ \frac{\sin(\omega t) - \omega t}{\omega^2} \left[\left(\sum_{l=1}^{L} i_l\right)^2 - \left(\sum_{l=1}^{L} j_l\right)^2\right]\right.\right.$$

$$\left.\left. + i\frac{1-\cos(\omega t)}{\omega^2}\left[\sum_{l=1}^{L}(i_l - j_l)\right]^2\right\}\right\}$$

(14)

$$\times \exp\left\{-\left[\sum_{l=1}^{L}(i_l - j_l)\right]\int d\omega \kappa(\omega) \frac{e^{i\omega t}-1}{\omega}a_\omega^+(0)\right\}\rho_{\{i_l,j_l\}}(0)$$

$$\times \exp\left\{\left[\sum_{l=1}^{L}(i_l - j_l)\right]\int d\omega \kappa(\omega) \frac{e^{-i\omega t}-1}{\omega}a_\omega(0)\right\}.$$

The operator $\rho_{\{i_l,j_l\}}(t)$, so $\rho_{\{i_l,j_l\}}(-t)$, has been expressed by $\rho_{\{i_l,j_l\}}(0)$ and $a_\omega(0)$. Therefore, $\langle\{\alpha_\omega\}|\rho_{\{i_l,j_l\}}(-t)|\{\alpha_\omega\}\rangle$ is obtained as a function of $\rho_{\{i_l,j_l\}}(0)$ and α_ω. Substituting this result into Eq. (9), we thus have

$$\rho_s(t) = \sum_{\{i_l,j_l\}} c_{\{i_l,j_l\}} \exp\left\{-\eta(t)\left[\sum_{l=1}^{L}(i_l - j_l)\right]^2\right\}$$

(15)

$$\cdot \exp\left\{i\Delta\phi(t)\left[\left(\sum_{l=1}^{L} i_l\right)^2 - \left(\sum_{l=1}^{L} j_l\right)^2\right]\right\} \cdot \rho_{\{i_l,j_l\}}(0),$$

where the Lamb phase shift factor $\Delta\phi(t)$ is defined as

$$\Delta\phi(t) = \int d\omega \kappa^2(\omega)\left[\frac{\omega t - \sin(\omega t)}{\omega^2}\right]$$

(16)

and the phase damping factor $\eta(t)$ is

$$\eta(t) = \int d\omega \kappa^2(\omega) \frac{4\sin^2\left(\frac{\omega t}{2}\right)}{\omega^2}\left(\langle N_\omega\rangle + \frac{1}{2}\right).$$

(17)

Both the Lamb phase shift and the phase damping have contributions to decoherence of the qubits. The Lamb phase shift was missed in Ref. [20]. By examining the calculation there, we find there is a mistake. In Eq. (13) of Ref. [20], the time evolution operator $U(t)$ in the interaction picture is expressed as

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$U(t) = \exp\left[-\frac{i}{\hbar}\int_0^t H_I\left(t'\right)dt'\right]$. But this expression is not correct since there $\left[H_I(t), H_I\left(t'\right)\right] \neq 0$. Because of this error, a phase factor was missed in the evolution operator $U(t)$. This phase factor finally results in the Lamb phase shift. From Eq. (15), we see, in the case of collective decoherence, the Lamb phase shift does not reduce to zero. Eq. (17) shows that the phase damping is directly proportional to the mean photon number. At high temperature, i.e., $k_B T >> \hbar\omega$, decoherence is mainly induced by the phase damping. But at low temperature, the Lamb phase shift is of the same order of magnitude as the phase damping and it can not be neglected.

The state fidelity has been introduced to describe stability of quantum information [24]. For a pure input state $|\Psi(0)\rangle$, the fidelity is defined as

$$F = \langle\Psi(0)|\rho_s(t)|\Psi(0)\rangle = tr\left[\rho_s(0)\rho_s(t)\right], \tag{18}$$

where $\rho_s(0) = |\Psi(0)\rangle\langle\Psi(0)|$ and $\rho_s(t)$ indicates the output density operator of the system. Here we use the fidelity to describe decoherence of the qubits. Suppose the initial state of the qubits is expressed as $|\Psi(0)\rangle = \sum_{\{i_l\}} c_{\{i_l\}}|i_l\rangle$, where $|i_l\rangle$ with $i_l = \pm 1$ may represent the states $\left|\pm\frac{1}{2}\right\rangle$ of a spin-$\frac{1}{2}$ electron, or the states $|e\rangle$ and $|g\rangle$ of a two-level atom. From Eq. (15), we get the fidelity

$$F = \sum_{\{i_l, j_l\}} \left|c_{\{i_l\}}\right|^2 \left|c_{\{j_l\}}\right|^2 \exp\left\{-\eta(t)\left[\sum_{l=1}^{L}(i_l - j_l)\right]^2\right\}$$

$$\cdot \exp\left\{i\Delta\phi(t)\left[\left(\sum_{l=1}^{L} i_l\right)^2 - \left(\sum_{l=1}^{L} j_l\right)^2\right]\right\} \tag{19}$$

In the derivation, the following relations are used.

$$tr\left(\rho_{1,-1}\rho_{-1,1}\right) = tr\left(\rho_{-1,-1}^2\right) = tr\left(\rho_{1,1}^2\right) = 1, \tag{20}$$

$$tr\left(\rho_{1,1}\rho_{1,-1}\right) = tr\left(\rho_{1,1}\rho_{-1,1}\right) = tr\left(\rho_{-1,-1}\rho_{1,-1}\right)$$

$$= tr\left(\rho_{-1,-1}\rho_{-1,1}\right) = tr\left(\rho_{1,1}\rho_{-1,-1}\right) = 0. \tag{21}$$

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Eqs. (15) and (19) suggest, the qubits are decohered collectively. This is an interesting phenomenon. The phase shift and the phase damping are directly proportional to the factor $\sum_{l=1}^{L} (i_l - j_l)$. If the input density operator satisfies $\sum_{l=1}^{L} (i_l - j_l) = 0$, at any time the fidelity $F = \left[\sum_{\{i_l\}} \left| c_{\{i_l\}} \right|^2 \right]^2 = 1$ and the reduced density operator of the qubits $\rho_s(t) = \sum_{\{i_l,j_l\}} c_{\{i_l,j_l\}} \rho_{\{i_l,j_l\}}(0) = \rho_s(0)$. So no decoherence occurs at all even if the qubits are interacting with the environment. The states satisfying the condition $\sum_{l=1}^{L} (i_l - j_l) = 0$ are called the coherence-preserving states. Consider the following states

$$|\Psi_m(0)\rangle = \sum_{\{i_l\} \in A_m} c_{\{i_l\}} |\{i_l\}\rangle, \tag{22}$$

where m is a definite number and A_m denotes the set $\left\{ i_l \left| \sum_{l=1}^{L} i_l = m \right. \right\}$. Obviously, the relation $\sum_{l=1}^{L} (i_l - j_l) = 0$ is satisfied for this kind of states. So all the states (22) are the coherence-preserving states. In these states, the qubits are entangled with each other.

It is interesting to compare the collective decoherence with the independent decoherence. If the qubits couple independently to separate environments, similar to the derivation of Eq. (15), it is not difficult to obtain that at time t the reduced density $\rho_s'(t)$ of the qubits is expressed as

$$\rho_s'(t) = \sum_{\{i_l,j_l\}} c_{\{i_l,j_l\}} \exp\left[-\eta(t) \sum_{l=1}^{L} (i_l - j_l)^2 \right] \cdot \exp\left[i\Delta\phi(t) \sum_{l=1}^{L} (i_l^2 - j_l^2) \right] \cdot \rho_{\{i_l,j_l\}}(0)$$

$$= \sum_{\{i_l,j_l\}} c_{\{i_l,j_l\}} \exp\left[-\eta(t) \sum_{l=1}^{L} (i_l - j_l)^2 \right] \cdot \rho_{\{i_l,j_l\}}(0). \tag{23}$$

The Lamb phase shift reduces to zero for the independent decoherence since for $i_l = \pm 1$ we always have $i_l^2 - j_l^2 = 0$. Eq. (23) shows that the phase damping increases with L (number of the qubits) monotonically. In general, $\rho_s'(t)$ rapidly

deviates from $\rho'_s(0)$ if L is large. This can be clearly seen from the state fidelity (indicated by F')

$$F' = \sum_{\{i_l, j_l\}} \left|c_{\{i_l\}}\right|^2 \left|c_{\{j_l\}}\right|^2 \exp\left[-\eta(t) \sum_{l=1}^{L} (i_l - j_l)^2\right] \qquad (24)$$

The typical behavior of F' in the form of Eq. (24), as has been discussed in Ref. [9], is $F' \propto e^{-\alpha(t)L}$, i.e., the fidelity decays with L exponentially. Its damping is insensitive to the type of the initial states. This is much different from decoherence of the qubits coupling to the same environment. In the latter case, with some input states decoherence of the qubits may increase with L more rapidly. But with some other states, i.e., the coherence-preserving states, no decoherence occurs at all. Sensitivity to the type of the input states in an important property of the collective decoherence.

To reduce the independent decoherence, many quantum error correction schemes have been proposed [11-17]. The schemes in Ref. [11-16] are devised to correct single qubit errors. In practice, one need repeatedly use these schemes to correct errors. For the independent decoherence, it can be shown easily that if N error corrections are performed within a time interval $[0, T)$, there is a remaining error probability of order $O\left((T/N)^2\right)$ after each error correction event [11]. Thus the accumulated error at time T is of order $O\left(N(T/N)^2\right)$. This error can be made arbitrarily small by choosing a sufficient large N. However, this analysis does not hold for the collective decoherence, since in the latter case occurrence of errors for different qubits is correlated. In fact, the error correction schemes are not very efficient for reducing the collective decoherence, since they do not take into account the specific interaction properties between the qubits and the environment. Fortunately, for reducing the collective decoherence, there is a simple and more efficient scheme. This scheme essentially exploits the coherence-preserving

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states. Before storing a state into the memory, we transform it into a coherence-preserving state in the form of Eq. (22). The transformed state undergoes no decoherence in the noisy memory and afterwards, it can be transformed back into the original state. In this scheme we should find a one-to-one map from arbitrary input states onto the coherence-preserving states in a larger Hilbert space. Suppose there are $2L$ qubits. The Hilbert space spanned by the coherence-preserving states (22) with $\{i_l\} \in A_0$ is indicated by S_0. The dimension of S_0 is $\begin{pmatrix} 2L \\ L \end{pmatrix}$. If all the states in the space S_0 are efficiently used in the transformation, the efficiency η_m of this scheme attains

$$\eta_m = \frac{1}{2L} \log_2 \begin{pmatrix} 2L \\ L \end{pmatrix} \approx 1 - \frac{1}{4L} \log_2 (\pi L). \tag{25}$$

The approximation is taken under the condition $L >> 1$. So the maximum efficiency is near to 1 if L is large. Of course, to make use of all the states in S_0, it requires an involved encoding. A simple encoding , though it is not the most efficient, is to use two qubits to encode one qubit. As has been mentioned in Ref. [20], the encoding is

$$\begin{aligned} |+1\rangle &\longrightarrow |+1, -1\rangle, \\ |-1\rangle &\longrightarrow |-1, +1\rangle. \end{aligned} \tag{26}$$

This encoding makes use of a subset of the coherence-preserving states in S_0. The encoding (26) can be easily fulfilled by using the quantum controlled-NOT gates [25].

In this paper, the coherence-preserving states are obtained with the assumption that the qubits in the memory undergo no amplitude damping. If the amplitude damping in not negligible, the states (22) will not remain unchanged. However, in Ref. [25], we developed a general method to set up the coherence-preserving states. By a strategy called the free Hamiltonian elimination, the coherence-preserving states are found to exist both for the phase damping and for

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the amplitude damping, though they are not of the same form. Furthermore, we shown there that the coherence-preserving states could be operated on with quantum gates. These results suggest, the transformation to the coherence-preserving states is a useful and efficient scheme for reducing the collective decoherence.

The Hamiltonian (1) also describes decoherence of a spin-$\frac{1}{2}$ chain. The spin chain can be adopted as a model of apparatus in some case. Therefore, the decoherence model in this paper may also have some implications for quantum measurements. In fact, the coherence-preserving states discussed above are intimately related to the concept of the point basis introduced by Zurek some years ago [21]. It has been recognized that decoherence plays an essential role in quantum measurements [26]. Decoherence is induced by the inevitable interaction between the apparatus and the environment. This decoherence causes the off-diagonal terms of the density operator to decay in the point basis of the apparatus and leads to the wave-packet collapse. The point basis consists of the eigenvectors of the operator which commute with the apparatus-environment interaction Hamiltonian. The coherence-preserving states just have this property, so they make a point basis. It is nice to see how fast the off-diagonal terms of the density decay in the point basis. To show this, we need to analyze the time behaviors of $\eta(t)$ and $\Delta\phi(t)$. At high temperature, $\eta(t)$ is much more important. For the one-dimensional spin chain, $\kappa^2(\omega)$ has the form of $\kappa^2(\omega) = \varepsilon^2\omega/\hbar$, where ε is approximately a constant [9]. In the high temperature limit, Eq. (17) gives

$$\eta(t) \approx \frac{\pi\varepsilon^2 k_B T}{\hbar^2} t. \tag{27}$$

The decoherence time is thus $\frac{\hbar^2}{\pi\varepsilon^2 k_B T}$. Comparing it with the decoherence time for harmonic oscillators, we see, for the spin chain, the decoherence time follows a similar dependence on various parameters (such as the coupling constant and

temperature of the environment) as is the case for harmonic oscillators.

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